[50th Anniversary Symposium]

The Mathematical and Numerical Construction of Turbulent Solutions for the 3D Incompressible Euler Equation and Its Perspectives

*Uriel Frisch, Laboratoire Lagrange, Observatoire de la Cote d'Azur Universite Cote d'Azur, Nice, France
(in collaboration with Laszlo Szekelyhidi, Department of Mathematics, University of Leipzig, Germany
and Takeshi Matsumoto, Department of Physics, Kyoto University, Japan)
*E-mail for correspondence: uriel@oca.eu

Starting with Kolmogorov's 1941 (K41) work [1], infinite Reynolds number flow is known to have velocity increments over a small distance r that vary roughly as the cubic root of r. Formally, such flow is expected to satisfy Euler's partial differential equation, but the flow being not spatially differentiable, the equation is satisfied only in a distributional sense. Since Leray's 1934 work [2], such solutions are called weak. Actually they were already present –very briefly– in Lagrange's 1760/1761 work on non-smooth solutions of the wave equation [3].

A major breakthrough has happened recently: mathematicians succeeded in constructing rigourously weak solutions of the Euler equation whose spatial regularity –measured by their Hölder continuity exponent– is arbitrarily close to the value predicted by K41 (Isett 2018 [4], Buckmaster et al. 2017 [5]). Furthermore these solutions present the anomalous energy dissipation investigated by Onsager in 1949 (Ons49).

We shall highlight some aspects of the derivation of these results which took about ten years and was started originally by Camillo de Lellis and Laszlo Szekelyhidi and continued with a number of collaborators. On the mathematical side the derivation makes use of techniques developed by Nash (1954) for isometric embedding [6] and by Gromov (1986, 2017) for convex integration [7, 8]. Fortunately, many features of the derivation have a significant fluid mechanical content. In particular the successive introduction of finer and finer flow structures, called Mikados by Daneri and Szekelyhidi (2017) [9], because they are slender and jetlike. The Mikados generate Reynolds stresses on larger scales; they can be chosen to cancel discrepancies between approximate and exact solutions of the Euler equation.

A particular engaging aspect of the construction of weak solutions is its flexibility. The Mikados can be chosen not only to reproduce K41/Ons49 selfsimilar turbulence, but also to synthesize a large class of turbulent flows, possessing, for example, small-scale intermittency and multifractal scaling. This huge playground must of course be explored numerically for testing all manners of physical phenomena and theories, a process being started in a collaboration between Leipzig, Nice, Kyoto and Rome.

References

1) Kolmogorov, A. N.: *Dokl. Akad. Nauk SSSR*, 30 (1941) 209-303.
2) Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, 63 (1934) 193-248.
3) Lagrange, J. L.: Nouvelles recherches sur la nature et la propagation du son, *Misc. Taurinensia*, 2 (1760/1761), Oeuvres, 1. pp. 151-332 (see Section 5).
4) Isett, P.: A proof of Onsager's conjecture, *Annals Math.*, in press (http://annals.math.princeton.edu/articles/12779).
5) Buckmaster, T., De Lellis, C., Szekelhyhidi, L. Jr., Vicol, V.: Onsager's conjecture for admissible weak solutions, (2017) arXiv:1701.08678 [math.AP].
6) Nash, J.: C^1 isometric imbeddings, *Ann. Math.*, 60 (1954) 383-396.
7) Gromov, M.: *Partial Differential Relations* (Springer, 1986).
8) Gromov, M.: Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems, *Bull. Amer. Math. Soc.*, 54 (2017) 173-245.
9) Daneri, S., Szekelhyhidi, L. Jr.: Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations, *Arch. Rat. Mech. Anal.*, 224 (2017) 471-514.