[50th Anniversary Symposium]

Part I. Landscape of JSFM 50 years ago; Part II. New perspectives on mass conservation law and waves in fluid mechanics

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Abstract: The Japan Society of Fluid Mechanics originated from a preliminary organization consisting of pioneer researchers of fluid mechanics in 1968, fifty years ago, and developed into a formal academic society in 1982. The first part of this presentation is devoted to recollecting scientific achievements in Japan made by the pioneers already at the time of its start. Their impacts on fluid mechanics are shining still now. In the second part, from purely scientific and modern point of views, we focus on physical aspects of mass conservation law and waves in fluid mechanics.

1 Introduction

Fundamental conservation equations of fluid mechanics are derived as a non-relativistic limit from the relativistic fluid mechanics[1]. The relativistic energy equation can be decomposed in the following way (Ref.[2], Appendix B.4):

$$\Big(\underline{\text{Rest mass}}\text{ part of }O(c^2)\Big) + \Big(\underline{\text{Flow energy}}\text{ part of }O(u^2)\Big) = 0,$$

where c is the light speed and u a representative magnitude of fluid velocity $\boldsymbol{v}$. In the non-relativistic limit $u/c \to 0$, this equation splits into two parts. From the first term, the so-called continuity equation is deduced, while from the second term the well-known energy conservation equation is derived. Thus in the non-relativistic limit, we obtain

$$\partial_t \rho \;+\; \mathrm{div}(\rho \boldsymbol{v}) = 0, \tag{1}$$

$$\partial_t \Big(\rho\big(\tfrac{1}{2}\,v^2 + \epsilon\big)\Big) \;+\; \mathrm{div}\Big(\rho \boldsymbol{v}\big(\tfrac{1}{2}\,v^2 + h\big)\Big) = 0\,, \tag{2}$$

where ρ is the fluid density, ϵ the internal energy and h the specific enthalpy. Here we confront unusual situation. From a single relativistic energy equation related to the symmetry of time-translation invariance, we have two conservation equations in the limit, $u/c \to 0$. However, the Noether's theorem[3] of theoretical physics states, "*Symmetries imply conservation laws*".[4] Are the above results of nonrelativistic limit satisfactory ? Or should we seek another way to resolve it ? We will revisit this issue in Part II. In Part I, we recollect the research-landscape of fluid mechanics community in Japan fifty years ago or earlier.

Part I: Landscape of JSFM fifty years ago

2 Pioneering studies before the start of JSFM

Before the start of our society JSFM, one can recognize significant scientific achievements already made by a number of leading members of the society.

(a) *Laminar viscous flow around a circular cylinder*

One of the most significant achievements at those early times was the successful analytical study of the late Professor Isao Imai (1951)[5], who investigated far field of the viscous laminar flow around a circular cylinder. This was a successive approximation to the Navier-Stokes (NS) equation, starting from the Oseen's solution valid asymptotically at great distances. This is regarded as an external part of the whole flow field. An internal field matching to this external field and satisfying the non-slip boundary condition over the cylinder surface was given numerically by M. Kawaguti (1953)[6] at the Reynolds number $R = 40$ (based on the cylinder diameter) with using a hand-calculator (Tyger). This was one of the earliest DNS's of the NS equation, except Thom (1933, *Proc. Roy. Soc.* A141, 651: with R =10, 20). The streamlines thus obtained were compared with the visualization experiment by S. Taneda (1956)[8], showing a pair of standing eddies in the wake of the cylinder. Agreement was very excellent. The numerical achievement of Kawaguti paved a path to go ahead to develop an analytical scheme finding uniformly valid solution of the NS equation over the whole field. The present speaker has a strong feeling[9] that the set of three works gave a stimulating hint for later (around 1957) development of the method of *Matched Asymptotic Expansions*. In addition, their combined works have provided strong evidence that the NS equation can describe steady laminar flows at moderate Reynolds numbers observed experimentally.[6]

(b) *Stability of laminar flows and turbulence*

Stability of laminar shear flows were studied both math-

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ematically and experimentally from fifties to sixties. According to the linear stability theory, a two-dimensional jet was studied by Tatsumi & Kakutani (1958)[11], and free shear layers by Tatsumi & Gotoh (1960)[12]. Experimental study of a two-dimensional jet was carried out by Sato (1960)[13] on its stability and transition to turbulence.

First mathematical study in Japan on turbulence according to the statistical theory was made by Prof. T. Tatsumi (1957)[10], investigating initial decay of homogeneous isotropic turbulence by assuming *zero fourth-order-cumulant* of incompressibile velocity field.

(c) *Streamwise vortices in boundary layer flows*

At the time of sixties, there was a gap between the observed phenomena of boundary layer transition to turbulence and the stability analyses which were mainly concerned with linear study of 2D disturbance waves. Formation of 3D-disturbances was regarded as prerequisite for the flow transition to turbulence in the boundary layer. An essential role of *streamwise* vortices was recognized for creation of three-dimensionality in the boundary layer. This transition problem was reviewed by the late Professor Itiro Tani (1969)[16], and studied by Tani & Komoda (1962)[14], collaborating with the late Prof LSG Kovasznay staying in Tokyo.* The vortices cause *redistribution* of mean velocity field. Later, the *streak* structure in boundary layer flows was interpreted by this mechanism.

(d) *Nonlinear waves*

A blast wave is usually generated as a shock-incident caused by a powerful explosion such as a supernova or an atomic bomb. The velocity U within the blast wave is not constant and always larger than the sound speed c_s. Certain exact solution of the blast wave problem was presented by A. Sakurai (1955)[7] for each of spherical, cylindrical and planar symmetry. citing the paper of G.I. Taylor:[17].

In regard to fluid motions caused by locally concentrated vorticity, one of the well-known laws is the *local-induction* law.† In regard to the local-induction law associated with concentrated vorticity, a soliton formulation was proposed by Hasimoto (1972)[19], who transformed the the law into the nonlinear Schrödinger equation.

(e) *International relation and collaboration*

International communications were carried out with both ways of receiving foreign scientists and oversea-visit of Japanese scientists. In 1928, the Kawanishi Aircraft (Co.) in Kobe invited Theodore von Karman (of age before 50) from RWTH Aachen of Germany, who designed a big wind tunnel of outflow diameter 2 m and its maximum flow speed 45 m/sec. Soon after the Japan stay, he was appointed as the director of the Guggenheim Aeronautical Laboratory at the California Institute of Technology (GALCIT). In 1929, Ludwig Prandtl (at the age 54) was invited to the Aeronautical Research Institute (of Tokyo Imperial University) and gave three-day lecture at the Institute. In his lecture, he showed two pictures of visualization comparing boundary layer separations over the surface of a sphere with and without a trip wire. The trip wire caused a dramatic effect on the boundary layer separation owing to transition to turbulent boundary layer.

Late Professor Susumu Tomotika visited United Kingdom for two years from 1934 and studied *Fluid-Dynamics* under Sir Geoffrey Taylor. At this time he was a Professor at the *Osaka Imperial University*, and Imai joined him from Tokyo as a research assistant in 1936. In 1938, Tomotika was appointed as Professor at Kyoto Imperial University, and Imai returned to Tokyo as a lecturer. In Kyoto, Tatsumi was appointed as a research assistant of Tomotika's group in 1946, and later as an associate professor.

From 1955 to 1957, Imai stayed at Maryland University and Cornell University in USA by invitation in view of his great contributions with exact mathematical analyses in both fields of viscous flows and high-speed flows. During his stay, he met a number of well-known scientists: Th. von Karman, G.K. Batchelor, J.M. Burgers. Furthermore, he visited Johns Hopkins, New York and Boston and met F. Clauser, L.S.G. Kovasznay, G.B. Schubauer, P.S. Klebanoff, S. Goldstein, C.C. Lin, and W.R. Sears. Itiro Tani was also staying at Caltech in 1959 to 60. There remains a photograph[20] showing von Karman and Tani together at their seminar meeting.

In 1960, there was IUTAM Symposium "*Magnetohydrodynamics*" held at Williamsburg in USA, where there were several Japanese participants, in addition to Tani and Imai, also Tatsumi, Hasimoto and others. Since then, the 1960s was an age of MHD in Japan. The IUTAM is an abbreviation of *International Union of Theoretical and Applied Mechanics*, to which our country joined in 1951.

There was a section of *Fluid Physics* at JPL (Jet Propulsion Laboratory of NASA) administrated by Caltech. Besides its work in rocket propulsion, JPL was an impor-

*The peak-valley structure associated with the streamwise vortices reviewed by Tani was visualized by Hino *et al.*[15] with *hydrogen bubbles*.

†When a vortex filament is distorted slightly from its circular form, the deformation wave moves along the filament like a travelling wave. This was observed by Kambe & Takao (1971)[18] in smoked vortices.

tant aeronautical laboratory with a supersonic wind tunnel, where supersonic boundary layers and instability waves were studied. In 1960's, they received Japanese visitors: T, Tatsumi, H. Sato, and H. Komoda.

Fifty-two years ago (1966) at Kyoto, there was held an international Symposium IUGG-IUTM, in which a number of renowned scientists participated (see Fig.1). After the Kyoto conference, George Batchelor visited Taneda's laboratory at the Institute of Applied Mechanics, Kyushu Univ., and got interested in various visualization experiments carried out there by S. Taneda (1956), and also by Okabe & Inoue (1960, 61). Batchelor cited a number of photographs of their visualization in his textbook[21)].

By the Scholarship of British Council, Prof. Tatsumi stayed at University of Cambridge in early 1950's. His stay was helped by G.I. Taylor and George Batchelor. Present speaker (T. Kambe) also stayed at Darwin College of Cambridge Univ. (1974-75) by the B-C Scholarship, and learned the authentic style of Fluid Mechanics under George Batchelor (with respect) and Sir James Lighthill.

Part II. New perspectives on fluid mechanics

In this new approach, the mass conservation equation plays a vital role. Traditional formulation regards the mass conservation law *a priori* valid, and does not try to relate it with any physical symmetry. We review such an approach first in §3.1, and consider symmetry issues in §3.3.

3 Flows of an ideal fluid

3.1 Symmetry and conservation law

Let us consider the variational formulation for flows of an ideal fluid. The action S and Lagrangian $\Lambda(\boldsymbol{v}, \rho, s)$ are[22)]

$$S = \int \Lambda(\, q_\lambda(x^\mu)\,)\,\mathrm{d}\Omega, \quad \Lambda(q_\lambda) \equiv \tfrac{1}{2}\,\rho \boldsymbol{v}^2 - \rho\epsilon(\rho), \quad (3)$$

where $\mathrm{d}\Omega = \mathrm{d}t\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, and $q_\lambda(x^\mu)$ ($\mu = 0,1,2,3$; $\lambda = 1,\cdots$) denote $\boldsymbol{v} = (v^1, v^2, v^3)$ and ρ, where $s = const$ (or $\mathrm{d}s = 0$) is assumed.[†] The Lagrangian density Λ can be written also as $\Lambda(X^k_\mu, X^k) = \frac{1}{2}\, X^k_0\, X^k_0 - \epsilon(X^k_l, X^k)$ by the *Lagrangian description* for a particle position $X^k(\alpha^\mu)$ with the time $\alpha^0 = t$, $\alpha^k = X^k(t=0)$, and $X^k_\mu = \partial X^k/\partial\alpha^\mu$ for $k, l = 1,2,3$. The Euler-Lagrange equation is obtained by requiring the Λ-variation resulting from arbitrary variational transformation $X^k \to X^k + \delta X^k$ to vanish, *i.e.* $\delta\Lambda = 0$:

$$(\mathcal{L}_{exp})_k \equiv \frac{\partial}{\partial\alpha^\mu}\Big(\frac{\partial\Lambda}{\partial X^k_\mu}\Big) - \frac{\partial\Lambda}{\partial X^k} = 0; \quad k = 1,2,3. \ (4)$$

This is obtained under vanishing conditions of boundary values. One can define an energy-momentum tensor T^ν_μ by

$$T^\nu_\mu \equiv X^k_\mu \Big(\frac{\partial\Lambda}{\partial X^k_\nu}\Big) - \Lambda\,\delta^\nu_\mu\,. \quad (5)$$

Fluid is assumed *incompressible*: $\det(X^k_l) = 1$. Taking simple variation of Λ (without boundary consideration and using $\partial_\nu \equiv \partial/\partial\alpha^\nu$), we obtain the following equality,[3,22)]

$$[\mathcal{L}_{exp}]_k\,\delta X^k = [\partial_\nu T^\nu_\mu]\,\delta X^\mu,$$

which vanishes by (4). This is the *Noether's theorem*[3)]. Namely, if the equation (4) is satisfied, we obtain the conservation law, $\partial_\nu T^\nu_\mu = 0$. This yields four conservation equations of energy ($\mu = 0$) and momentum components ($\mu = 1,2,3$). This is a Mechanics version of "*Symmetries (i.e. invariances of Λ) imply conservation laws*", noted in the Introduction. In this Lagrangian description, the mass conservation law is simply $\partial_t(\mathrm{d}V_\alpha) = 0$ where $\mathrm{d}V_\alpha \equiv \mathrm{d}\alpha^1\mathrm{d}\alpha^2\mathrm{d}\alpha^3$.

The variational principle with the *Eulerian description* yields the following Euler-Lagrange equation (§7.5, Ref.17)) :

$$\frac{\partial}{\partial t}\Big(\frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^l}\Big(v^l\,\frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^k}\Big(\Lambda - \rho\,\frac{\partial\Lambda}{\partial\rho}\Big) = 0. \quad (6)$$

Euler's equation of motion results from this, as follows:

$$\partial_t \boldsymbol{v} \ + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \ + \ \rho^{-1}\nabla\, p = 0. \quad (7)$$

The mass conservation (1) must be added as a constraint.

3.2 Sound waves and vortex sound

Consider an inviscid flow generated by localized vorticity in unbounded fluid of uniform density ρ_0 and uniform entropy s_0 with the sound speed c_0. Sound speed is defined by $c_s = [\mathrm{d}p/\mathrm{d}\rho]^{1/2}$ where the pressure is $p = p(\rho, s_0) = p(\rho)$, and $\rho^{-1}\nabla\, p = \nabla h$ since $(1/\rho)\mathrm{d}p = \mathrm{d}h - T\mathrm{d}s = \mathrm{d}h$. Note the identity: $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \equiv \nabla\frac{1}{2}\,v^2 + \boldsymbol{\omega}\times\boldsymbol{v}$.

Now, one can define two vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ by

$$\boldsymbol{E} \equiv -\partial_t \boldsymbol{v} - \nabla\, h_s, \quad \boldsymbol{H} \equiv \boldsymbol{\omega} = \nabla\times\boldsymbol{v}, \quad (8)$$

where $h_s \equiv h + \frac{1}{2}\,v^2$. The variables $\boldsymbol{v}$ and h_s are analogous to the vector potential and scalar potential of electromagnetism. From these definitions of $\boldsymbol{E}$ and $\boldsymbol{H}$ only, following *fluid* Maxwell equations are derived immediately:

$$\nabla\cdot\boldsymbol{H} = 0, \qquad \nabla\times\boldsymbol{E} + \partial_t\boldsymbol{H} = 0\,, \qquad (9a,b)$$

$$\nabla\cdot\boldsymbol{E} = q, \qquad c_0^2\,\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = \boldsymbol{J}, \qquad (9c,d)$$

[†] Position vector is given by $\boldsymbol{r} = (x^1, x^2, x^3)$, and time by $t = x^0$.

Uniform state of a fluid is described by two parameters of thermodynamic variables such as fluid density ρ, pressure p, specific internal energy ϵ (*i.e.* per unit mass), specific entropy s or specific enthalpy h.

(Kambe 2010)[23], where $q = -\partial_t(\nabla\cdot\boldsymbol{v}) - \nabla^2 h_s$, and $\boldsymbol{J} = \partial_t^2\boldsymbol{v} + \nabla\partial_t h_s + c_0^2\nabla\times(\nabla\times\boldsymbol{v})$, and the *charge conservation* equation ($\partial_t\, q + \mathrm{div}\,\boldsymbol{J} = 0$) is easily confirmed. The equations $(9a, \cdot\cdot, 9d)$ are derived independently of the continuity (1) and Euler's equation (7). But, time evolution of q and $\boldsymbol{J}$ are to be determined by solving those equations.

The Euler equation (7) determines the vector $\boldsymbol{E} = \boldsymbol{\omega}\times\boldsymbol{v}$, expressed by $\boldsymbol{v}$ only since $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$. Using this, the equation $(9b)$ reduces to the vorticity equation:

$$\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = 0. \tag{10}$$

The source term q of $(9c)$ is also given in terms of $\boldsymbol{v}$:

$$q = \nabla\cdot(\boldsymbol{\omega}\times\boldsymbol{v}). \tag{11}$$

With a representative velocity u, one can define a representative Mach number of the flow by $M = u/c_0$ ($\ll 1$, assumed). Perturbation waves of small amplitude are governed by the following wave equation[23]:

$$(a_0^{-2}\partial_t^2 - \nabla^2)h_s' = S(\boldsymbol{x}, t), \quad S = \nabla\cdot\boldsymbol{E} + q', \tag{12}$$

where h_s' is a small deviation of h_s from unperturbed value, and q' is a source term of higher order. An equation for the acoustic pressure p' is found after some calculus as follows:

$$(a_0^{-2}\partial_t^2 - \nabla^2)p' = \rho_0\mathrm{div}(\boldsymbol{\omega}\times\boldsymbol{v}). \tag{13}$$

The source term on the right $\rho_0\mathrm{div}(\boldsymbol{\omega}\times\boldsymbol{v})$ states that dynamical motion of the vorticity $\boldsymbol{\omega}$ excites acoustic waves. The equation (13) is called the *equation of vortex sound*.[24] Thus, the Euler's system describes *longitudinal* waves, excited by rotational flows, and observed experimentally.

Note that the equation of sound waves was first derived by L. Euler (1759), just after having propsoed his equation of fluid motion together with his continuity equation. He wrote the wave equation in a classic form, $\mathrm{dd}y/dt^2 = (p_a/\rho)(\mathrm{dd}y/\mathrm{d}x^2)$, where $y(x,t)$ is the displacement of air particle located at x in undisturbed state and p_a the air pressure. This work was inspired by the letter from a young mathematician Lagrange and his first paper (1759).[25]

3.3 Symmetry issues

In §3.1, the governing equations of ideal-fluid flow have been derived from the invariance of Lagrangian density Λ with respect to local gauge transformations[22] (of three coordinates): $X^k \to X^k + \delta X^k(\alpha^\mu)$. The Λ is also invariant to time translation. From the four invariances with respect to four coordinate transformations, four conservation equations have been derived: one energy equation and three momentum equations. This is common, whether the motion is relativistic or non-relativistic. However in the present non-relativistic case, the mass conservation was a condition required *a priori*, while in the relativistic case the mass conservation expression is swallowed into the energy conservation equation as the rest-mass energy change. In other words, the single energy equation of the relativistic case splits into two for non-relativistic flows: the mass conservation equation and the energy equation of fluid motion.‡ In the next section, we consider how the mass conservation law is formulated on the basis of the symmetry concept.

4 Mass conservation and gauge symmetry

Let us introduce a new *EM*-like field described by 4-vector potential $a_\mu(x^\nu) = (\phi_a,\, -\boldsymbol{a})$ in the 4-space-time of fluid-flow.¶ In fact, the mass conservation law is closely associated with the gauge invariance of the new *EM*-like field. Let us define one-form $\mathcal{A}$ (a *gauge field*) by

$$\mathcal{A} = a_\mu \mathrm{d}x^\mu = \phi_a\,\mathrm{d}t - a_x\mathrm{d}x - a_y\mathrm{d}y - a_z\mathrm{d}z.$$

A pair of *fluid*-EM fields $\boldsymbol{e}$ and $\boldsymbol{b}$ are defined by

$$\boldsymbol{e} \equiv -\partial_t\boldsymbol{a} - \nabla\phi_a\,, \qquad \boldsymbol{b} \equiv \nabla\times\boldsymbol{a}\,, \tag{14}$$

Taking external differential d of $\mathcal{A}$, we obtain the *field strength* two-form $\mathcal{F} = \mathsf{d}\mathcal{A} = \sum F_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$:

$$(\,F_{\mu\nu}\,) = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -b_3 & b_2 \\ -e_2 & b_3 & 0 & -b_1 \\ -e_3 & -b_2 & b_1 & 0 \end{pmatrix}.$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu = -F_{\nu\mu}$. Using the field tensor $(F_{\mu\nu})$, one can derive *fluid Maxwell equations*: [2,26]

$$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} = 0, \qquad \partial_\mu G^{\mu\nu} = j^\nu, \tag{15}$$

where $j^\nu = (\rho, \boldsymbol{j})$ is a current 4-vector.§ The first gives $\nabla\cdot\boldsymbol{b} = 0$, and $\partial_t\boldsymbol{b} + \nabla\times\boldsymbol{e} = 0$, while the second gives

$$\nabla\cdot\boldsymbol{d} = \rho, \qquad -\partial_t\boldsymbol{d} + \nabla\times\boldsymbol{h} = \boldsymbol{j}. \tag{16}$$

If we make a *gauge transformation*: $\mathcal{A} \to \mathcal{A}' = \mathcal{A} - \mathsf{d}\psi$ (with ψ a scalar function), we find *gauge invariance* of $F_{\mu\nu}$. In fact, by using $a'_\mu = a_\mu - \partial_\mu\psi$, we have

$$F'_{\mu\nu} = \partial_\mu a'_\nu - \partial_\nu a'_\mu = \partial_\mu\big((a_\nu - \partial_\nu\psi) - \partial_\nu(a_\mu - \partial_\mu\psi)\big) = F_{\mu\nu}.$$

‡This reminds us of a phenomenon of *particle physics*, *that is* splitting of the unified electroweak-force into electromagnetic force and weak nuclear-forces, when the temperature drops from the huge temperature in the early Universe.

¶ *EM* : *Electro-Magnetism*. $x^\nu = (t, x^1, x^2, x^3)$ and $\boldsymbol{a} = (a^1,\, a^2,\, a^3)$. External differential d of a scalar field ψ is defined by $\mathsf{d}\psi = \partial_t\psi\,\mathrm{d}t + \partial_x\psi\,\mathrm{d}x + \partial_y\psi\,\mathrm{d}y + \partial_z\psi\,\mathrm{d}z$.

§ $(G^{\mu\nu})$ is defined with replacing $\boldsymbol{e}$ by $-\boldsymbol{d}$ and $\boldsymbol{b}$ by $\boldsymbol{h}$ from the matrix $(F_{\mu\nu})$, where $\boldsymbol{d} = \varepsilon\boldsymbol{e}$ and $\boldsymbol{h} = \lambda^{-1}\boldsymbol{b}$ with ε and λ parameters.

This is equivalent to the well-known gauge transformation of the *Electro-Magnetism*:[26] $\phi_a \to \phi_a - \partial_t \psi$, and $\boldsymbol{a} \to \boldsymbol{a} + \nabla\psi$, and importantly the EM-like fields $\boldsymbol{e}$ and $\boldsymbol{b}$ are *invariant* with this transformation. This property owes to the *anti-symmetry* of $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The $(\rho, \boldsymbol{j})$ on the RHS of (16) are gauge invariant too.

Let us take 4-divergence of the second equation of (15): $0 = \partial_\nu \partial_\mu G^{\mu\nu} = \partial_\nu j^\nu$. The double differential operation is symmetric: *i.e.* $\partial_\nu\partial_\mu = \partial_\mu\partial_\nu$, while the field strength tensor is anti-symmetric: $G^{\mu\nu} = -G^{\nu\mu}$. Hence the total sum $\partial_\nu\partial_\mu G^{\mu\nu}$ (for $\mu, \nu = 0, \cdots, 3$) vanishes. Thus we obtain the conservation equation for the current $\boldsymbol{j} = \rho\boldsymbol{v}$:

$$\partial_\nu j^\nu = \partial_t \rho + \partial_k (\rho\boldsymbol{v})_k = \partial_t\rho + \nabla \cdot (\rho\boldsymbol{v}) = 0. \quad (17)$$

The mass conservation property is closely related to the anti-symmetry of the tensors $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, which assures the *gauge* invariance of the *field strengths* $(\boldsymbol{e}, \boldsymbol{b})$ and the same invariance of $(\rho, \boldsymbol{j})$ by (16). Thus, the *mass conservation equation* is implied by the gauge invariance.

Note that the first of (16) accommodates a Coulomb-like force, $\boldsymbol{e} = -\nabla\phi_a$, which is not *external*. ¶

5 Combined fields of fluid flow and wavy field

Now, it is proposed[2)] that our system is a combination of two fields of fluid flow F and wavy field W (governed by the fluid Maxwell equations considered in the previous §4). According to a general principle of theoretical physics, such a combined field is defined by *linear* combination of Lagrangians describing each constituent field.‖

Equivalently, energy-momentum tenor $T^{\alpha\beta}_{\mathrm{fw}}$ of the combined field is linear combination of two tensors, $T^{\alpha\beta}_{\mathrm{f}}$ for the fluid flow and $T^{\alpha\beta}_{\mathrm{w}}$ for the wavy field. The system is governed by[2)]

$$\partial_\alpha T^{\alpha\beta}_{\mathrm{fw}} = \partial_\alpha T^{\alpha\beta}_{\mathrm{f}} + \partial_\alpha T^{\alpha\beta}_{\mathrm{w}} = 0\,, \quad (18)$$

without external excitation. Thus, we obtain the energy equation of the combined system by setting $\beta = 0$ as

$$\partial_t \big[\rho\big(\tfrac{1}{2}v^2 + \epsilon\big) + \tilde{e}_w\big] + \mathrm{div}\big(\boldsymbol{q}_f + \boldsymbol{q}_w\big) = 0\,, \quad (19)$$

where $\tilde{e}_w$ is the energy density of W-field, and $\boldsymbol{q}_f$ and $\boldsymbol{q}_w$ are energy fluxes. The momentum equation is given by $\partial_\alpha T^{\alpha k}_{\mathrm{f}} + \partial_\alpha T^{\alpha k}_{\mathrm{w}} = 0$ $(k = 1, 2, 3)$ for the combined system. Its 3-vector form is expressed by

$$\partial_t(\rho\boldsymbol{v} + \boldsymbol{g}) + \nabla\cdot(\boldsymbol{\Pi} + \mathbf{M}) = 0\,. \quad (20)$$

¶ The equation (21) includes the force $\rho\boldsymbol{e}$. With a scalar field $\phi_a = -gr^{-1}$, we have a gravity-like force: $-\rho\nabla\phi_a = \rho g \nabla r^{-1}$, from $\rho\boldsymbol{e}$.

‖Lagrangian of the wavy field is $\Lambda_w = -\frac{1}{4} F_{\alpha\beta}G^{\alpha\beta} + j^\beta a_\beta$.

The equation of fluid flow interacting with the wavy field is represented as $\partial_\alpha T^{\alpha\beta}_{\mathrm{f}} = -j_\lambda \overline{F}^{\lambda\beta}$ (Ref.[2]. Eq.43). Using this, the momentum equations for each of the fluid system and wavy system are written respectively as

$$\partial_t(\rho\boldsymbol{v}) + \nabla\cdot\Pi = \boldsymbol{F}_{\mathrm{L}}[\boldsymbol{a}]\,, \quad (21)$$

$$\partial_t\boldsymbol{g} + \nabla\cdot M = -\boldsymbol{F}_{\mathrm{L}}[\boldsymbol{a}]\,, \quad (22)$$

where $\boldsymbol{F}_{\mathrm{L}}[\boldsymbol{a}] = \rho\,\boldsymbol{e} + \boldsymbol{j}\times\boldsymbol{b}$ expressing the Lorentz-force-like interaction between the two field components, and $\Pi_{ij} = \rho v_i v_j + p\,\delta_{ij}$. Likewise, the energy equations for each fluid and wavy system are given by

$$\partial_t\big[\rho\big(\tfrac{1}{2}v^2 + \epsilon\big)\big] + \mathrm{div}\big(\boldsymbol{q}_f\big) = \boldsymbol{j}\cdot\boldsymbol{e}\,, \quad (23)$$

$$\partial_t\tilde{e}_w + \mathrm{div}\boldsymbol{q}_w = -\boldsymbol{j}\cdot\boldsymbol{e}\,. \quad (24)$$

The fluid internal energy ϵ can vary *thermodynamically* by absorbing heat liberated by a dynamical process if a dissipative mechanism is taken into account. If the gauge field a_μ of the W component vanishes, the equations (19) and (20) reduce to the familiar traditional equations, *i.e.* the energy equation and momentum equation of an ordinary fluid. The wavy field W assures the mass conservation by (17).

6 Concluding remarks

In physics, it is said that symmetries imply conservation laws. The law of mass conservation is one of the most fundamental laws of fluid mechanics. In traditional formulation, no appropriate argument is given about its physical symmetry with which the mass conservation is concerned. In this paper, the *gauge invariance* of the fluid Maxwell system (in §4) is shown to imply the law of mass conservation. As far as we have a law of current conservation, mathematics allows transversal wave-fields.

A new formulation of fluid mechanics[2)] is proposed by introducing a wavy-field Λ_{w} to the flow field Λ_{f}, and applied to turbulent streaky wall flows in the paper [2]. The details are to be presented at the lecture session. This theory can not only give appropriate description of transversal wave-field, but also be generalized so as to include a new dissipation mechanism comparable in magnitude with the *eddy viscosity* in turbulence.[2)] Self-contradiction is not incurred by this formulation. This formulation[2)] implies a new feature of the wall-bounded turbulence that the total velocity is expressed by a triple decomposition: (i) mean velocity $\boldsymbol{U}_m$, (ii) a wavy component $\boldsymbol{u}_w$, and (iii) turbulent component $\boldsymbol{u}'$. By taking account of the new mechanism of high rate of dissipation, the streaky structure of the wall turbulence is understood as a dissipative structure (by the paper [2]).

Figure 1: Joint-Symposium IUGG-IUTAM at Koto in 1966 (52 years ago, from Kobashi [20]). (**IUGG**: International Union of Geodesy and Geophysics; **IUTAM**: International Union of Theoretical and Applied Mechanics). *In the photo*, one can recognize (randomly): H. Görtler, F.N. Frenkiel, I. Tani, A. Roshko, A.M. Yaglom, L.S.G. Kovasznay, J.O. Hinze, M.T. Landahl, S.I. Pai, P.S. Klebanoff, G.K. Batchelor, M.J. Lighthill, P.G. Saffman, L.G. Loitsianski, R. Betchov, D.J. ,Benney, J. Laufer, and many Japanese participants.

The vortex sound is a longitudinal wave, while the Tollmien-Schlichting wave observed in viscous boundary layer flows is a transversal wave. The Euler's system is fitted to describing longitudinal waves. However, it appears that description of transversal waves is awkward in the Navier-Stokes system from physical point of view, because fluid incompressibility is always assumed for the analysis of transversal waves. It is expected that description of instability waves in viscous flows could be improved by the present framework taking account of mass conservation and waves.

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