Conservative and oscillation-less Semi-Lagrangian Schemes

Abstract: A type of semi-Lagrangian schemes was proposed to compute conservative transportation equation. The mass conservation and oscillation-suppressing properties are enforced by imposing proper constraining conditions to the reconstruction of interpolation function. Excellent numerical results were achieved for both linear and non-linear scalar conservation laws.

1. Introduction

Semi-Lagrangian methods that conserve exactly the transported physical field have been recently developed in [1,2] by using the CIP (Constrained Interpolation Profile) concept.

In this paper, we propose another class of schemes called CIP-CLS3 (Constrained Interpolation Profile - Conservative Semi-Lagrangian scheme with 3rd-order polynomial function). The CIP-CLS3 schemes are constructed from a cubic polynomial. In addition to the conservation constraint used in the CIP-CLS2, the slope (first-order derivative) of the interpolation function at the middle point of a mesh cell is also introduced as another constraint on the interpolation function. The slope at the cell center can be easily approximated from a reconstruction procedure and allows manipulations, i.e. slope limiters, to make the interpolation oscillation-less.

2. The schemes

The model equation to be considered is a transportation equation in one dimension

\[
\frac{\partial f}{\partial t} + \frac{\partial uf}{\partial x} = 0
\]  

where \( t \) refers to the time, \( x \) the spatial coordinate, \( u \) the characteristic speed and \( f \) the transported quantity.

We make use of a piecewise cubic polynomial function \( F_i(x) \).

As in the CIP-CLS2 method, a constraint for the conservation of cell-integrated average is imposed as

\[
\frac{1}{\Delta x_{i+1/2}} \int_{x_{i+1/2}}^{x_i} F_i(x) \, dx = \rho^*_{i+1/2}
\]

where \( \Delta x_{i+1/2} = x_i - x_{i+1} \).

Another constrained condition for the interpolation construction is imposed on the first-order derivative at the middle point of the cell as

\[
\frac{\partial F_i(x)}{\partial x} = \rho^*_{i+1/2}
\]

In terms of \( f_{i+1}^*, f_i^*, d_{i+1/2}^* \) and \( \rho^*_{i+1/2} \), the cubic polynomial \( F_i(x) \) can be completely determined. The numerical solution of \( f \) at time step \( n+1 \) is then updated via a semi-Lagrangian calculation.

The cell-integrated average \( \rho \) is advanced by the conservative relation

\[
\rho_{i+1/2}^* = \rho_{i+1/2}^* - (g_i - g_{i+1})
\]

where \( g \) represents the flux across the cell boundary during \( \Delta t \).

The slope of the interpolation function at the cell center \( d_{i+1/2}^* \) remains as a free parameter to be determined. It is this parameter that provides us a way to modify the interpolation function for suppressing numerical oscillation. Follows are two examples to define a CIP-CLS3 type scheme using conventional slope limiters.

The CIP-CLS3 UNO scheme:

We call a scheme CIP-CLS3 UNO if a UNO reconstruction[3] is used to evaluate the derivative \( d_{i+1/2}^* \). The CIP-CLS3 CW scheme:

In this scheme we adopt the approximation suggested by Colella and Woodward[4].

3. Numerical tests

We consider a one dimensional linear problem with an initial square profile. The computed results after 1000 steps of the CIP-CLS3 UNO and CIP-CLS3 CW are plotted in Fig.1. As expected, constrained by the oscillation suppressing reconstructions, both the schemes give well regulated and oscillation-less solutions. All the schemes conserve exactly the total mass.

![Fig.1 A transported square wave after 1000 step calculations with the CIP-CLS3 UNO scheme (left) and the CIP-CLS3 CW scheme (right).](image)

![Fig.2 Solution of Burgers' equation at t=0.75 with an initial condition given in [5]. Displayed are the results computed by the CIP-CLS3 UNO scheme (left) and the CIP-CLS3 CW scheme (right).](image)

The correct positions of the two shocks are obtained. The expansion waves are also accurately computed.

References


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