The mathematical and numerical construction of turbulent solutions for the 3D incompressible Euler equation and its perspectives

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$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \nu\nabla^2\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v} = 0 \quad \text{Euler/NS}$$

- Lagrange 1760 and the slow birth of weak solutions
- The Onsager 1949 conjecture on anomalous dissipation
- Scale-invariance vs. intermittency
- The Leipzig-Zurich school: proving the Onsager conjecture
- Towards weak dissipative and intermittent solutions: a world-wide collaboration

Lagrange 1760 and the slow birth of weak solutions

- As shown by D'Alembert, the vibrating string (wave) equation $\partial_{tt}\xi = \partial_{xx}\xi$ has the general solution $\xi = f(x-t) + g(x+t)$, where f and g are arbitrary.

- But what does this mean if the functions f and g are, say, discontinuous?
- Nowadays, we handle such a problem, using a distributional approach:
- We take a *test function* $\varphi(x,t)$, which is very smooth in both x and t
- We multiply the wave equation by φ, integrate over x and t and perform various integrations by parts, to obtain

$$\int dx \int dt\, \xi\, (\partial_{tt}\varphi - \partial_{xx}\varphi) = 0.$$

- This is, by definition, the *weak formulation* of the wave equation.
- This is roughly what Lagrange did in 1760/1761, except that his test function φ depended only on x and he could not eliminate $\partial_{tt}\xi$.
- Leray was the first to use the full space-time procedure on NS in 1934.

Onsager 1949 and anomalous dissipation

- In the twenties, Richardson, in his famous cascade poem (...*and so on to viscosity*) implicitly assumed a finite energy dissipation for NS in the limit $\nu \to 0$.

- In the early forties, Kolmogorov assumed a finite positive energy dissipation as $\nu \to 0$. [This was confirmed experimentally by Sreenivasan (1998) and numerically by Kaneda, Ishihara, Yokokawa, Itakura (2003).]

- Since the dissipation $\nu \int d\boldsymbol{x}\, |\nabla \boldsymbol{v}|^2$ is proportional to the viscosity, this is called *anomalous dissipation.*

- In the late forties, Onsager suspected that viscosity is not needed to obtain anomalous dissipation. Hence, one can just work with non-smooth solutions of the Euler equation.

- In 1949 Onsager stated that energy is conserved if the solution satisfies the condition that velocity increments over a small distance r are bounded by (const) r^α with $\alpha > 1/3$.

- But for $\alpha \le 1/3$, Onsager stated *that in principle, turbulent dissipation as described could take place just as readily without the final assistance of viscosity.*

It took nearly seventy years to prove this.

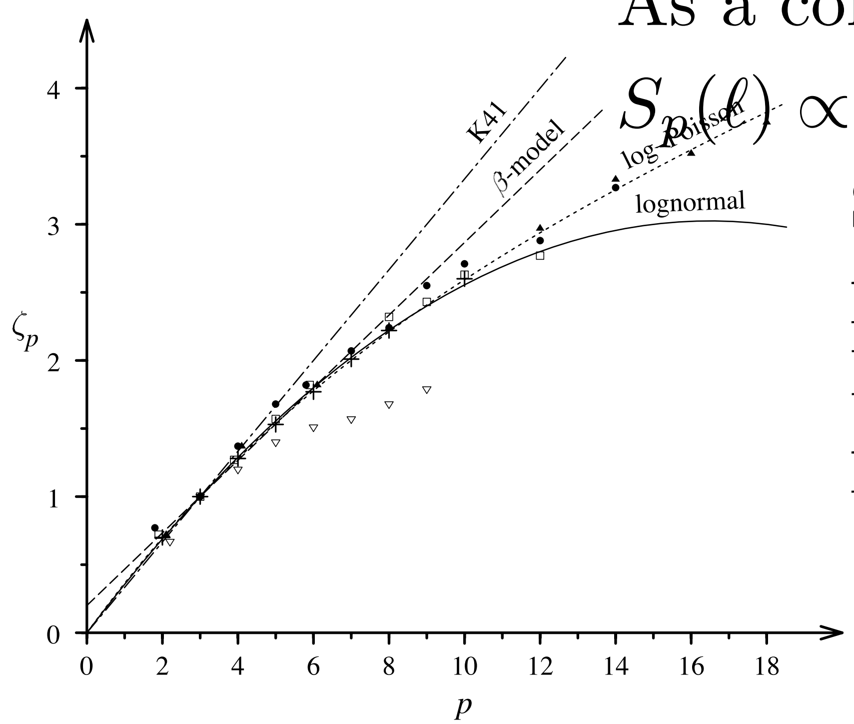
Scale invariance vs intermittency

- The Euler equation has a large set of invariance groups. Among these are the *scaling groups* $\boldsymbol{x} \to \lambda \boldsymbol{x}, \quad \boldsymbol{v} \to \lambda^h \boldsymbol{v}, \quad t \to \lambda^{1-h} t$ for arbitrary real h and positive λ.

- The Kolmogorov 1941 (K41) theory essentially assumes that, at infinite Reynolds numbers, statistical scale invariance holds. The scale-independence of the mean energy flux, then requires $h = 1/3$.

- As a consequence, *structure functions* $S_p(\ell) \equiv \langle \delta|\boldsymbol{v}(\ell)|^p \rangle$ satisfy $S_p(\ell) \propto \ell^{\zeta_p}$ with $\zeta_p = p/3$.

- Experimental and numerical evidence indicates that *scale invariance is actually broken.* The structure functions are indeed power laws in the ℓ, but the scaling exponent ζ_p is not a linear function of p. This suggests that the small-scale intermittency is *fractal*, as proposed by Mandelbrot, or even *multifractal*, as proposed by Parisi and Frisch.



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Simple phenomenological models of intermittency have been proposed, such as the β-model and the random β-model. However, it has never been shown that the Euler equation possesses actually fractal/multifractal solutions.

The mathematical and numerical construction of weak solutions (self-similar and beyond): the actors

- **Fifties - eighties: the forefathers**

  John Forbes Nash Jr. (1928 – 2015), Princeton. Isometric embedding theorem

  Misha Gromov. Paris, New York. Convex integration and h-principle for PDEs

- **2007 - 2017 Ten years of climbing the Onsager 1/3 peak**
  The convex integration approach

  Camillo De Lellis. Zurich, Princeton

  Tristan Buckmaster. New York, Princeton

  Sara Daneri. Leipzig, Erlangen

  László Székelyhidi. Leipzig

  Philip Isett. Princeton, Austin

  Vlad Vicol. Princeton

- **2017 — Numerical implementation and intermittency**

  Takeshi Matsumoto. Kyoto

  Greg Eyink. Baltimore

  Stefano Modena. Leipzig + **TB, LS**

  Luca Biferale. Roma

  Uriel Frisch. Nice

A Nash-type construction of weak Euler solutions

- **Goal**

  Construct in a time interval $[0, T]$ a (non-unique) 2π-periodic solution of the 3D Euler equation, whose velocity is spatially Hölder continuous of exponent $\alpha < 1/3$ and with a prescribed total energy function $E(t)$:

$$\partial_t \boldsymbol{v} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p, \quad \nabla \cdot \boldsymbol{v} = 0 \quad \text{Euler}$$

- **An inverse Renormalization Group strategy**

  with stages $\mathcal{S}_0, \mathcal{S}_1, \ldots \mathcal{S}_q, \ldots$ adding smaller and smaller-scale motions $(2\pi \to 2\pi/2, \to \ldots \to 2\pi/2^q \to \ldots)$.This gives rise to the usual Reynolds-averaged equations for $\langle \boldsymbol{v} \rangle_q \equiv \Sigma_{|\boldsymbol{k}|<2^q} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{v}}_{\boldsymbol{k}}$, namely

$$\partial \boldsymbol{v}_q + \nabla \cdot (\boldsymbol{v}_q \otimes \boldsymbol{v}_q) = -\nabla p_q - \nabla \cdot R_q, \qquad \nabla \cdot \boldsymbol{v}_q = 0,$$

$$R_q \equiv \langle \boldsymbol{v} \otimes \boldsymbol{v} \rangle_q - \boldsymbol{v}_q \otimes \boldsymbol{v}_q$$

  Eventually, for $q \to \infty$, the Reynolds stresses go (weakly) to zero.

- **Each stage consist of three successive steps**
  - Low-pass filtering step. Take the output of stage $(q-1)$ and apply $\langle \cdot \rangle_q$.
  - Euler-dynamical step (with gluing)
  - Mikado-perturbation step

Euler-dynamical step (with gluing)

- From the previous stage, we have an approximate solution of the Euler equation that has been low-passed filtered, killing all harmonics with wavenumbers $\geq 2^q$.
- We would like to improve this solution by letting it dynamically develop smaller-scale excitation.
- It is clearly enough for this to dynamically evolve the solution for about one small-scale eddy-turnover time.
- How is this done? Roughly, one samples the previous velocity field every turnover time. Then, one lets it evolve dynamically forward and backwards in time for about 2/3 of an eddy turnover time. Finally, one interpolates between the velocity fields in overlapping time intervals (gluing).

Important remark. Although this step involves solving various initial-value problems, they are always over short time intervals, for which one has good existence, uniqueness and regularity results. In contrast, the total time interval $[0, T]$ over which one constructs the weak solution can be much longer than that of good control over the behaviour of solutions. Globally we are not solving an initial-value problem, but proceeding *teleologically*.
目標指向の方法で

Mikado-perturbation step

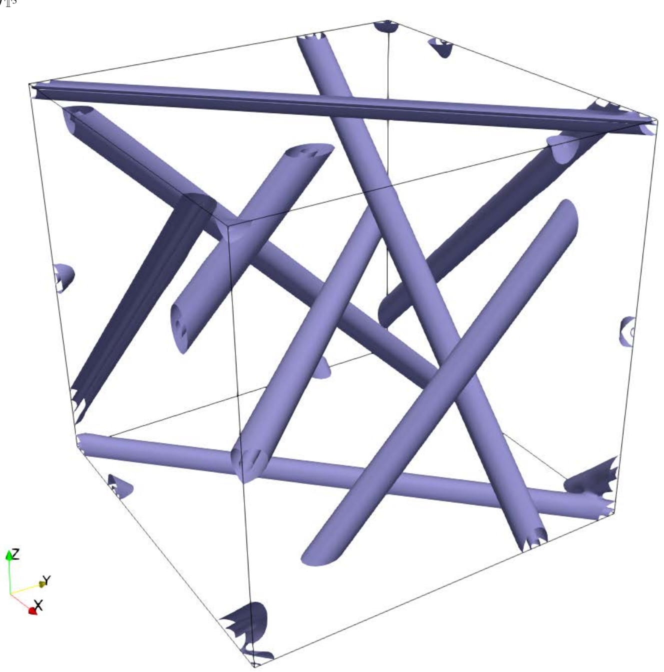
- We begin with a farfetched digression: how does one parallel park a car along the sidewalk [步道] of a street between two parked cars, leaving barely more than the space needed? Basically one wiggles in, performing tiny circular movements of smaller and smaller amplitude.

- This is roughly how Nash proceeded in his 1954 construction of isometric embeddings: in the successive stages he added transformations with harmonics of shorter and shorter wavelength. A similar device is used in the construction of weak Euler solutions: one adds, at stage q, six tiny cylindrical jets (called *Mikados*), whose radii vary as 2^{-q}. Note that the Reynolds stress tensor has six independent components. The six Mikado amplitudes can be adjusted to suppress the Reynolds stress error terms, while bringing the total energy closer to its desired value

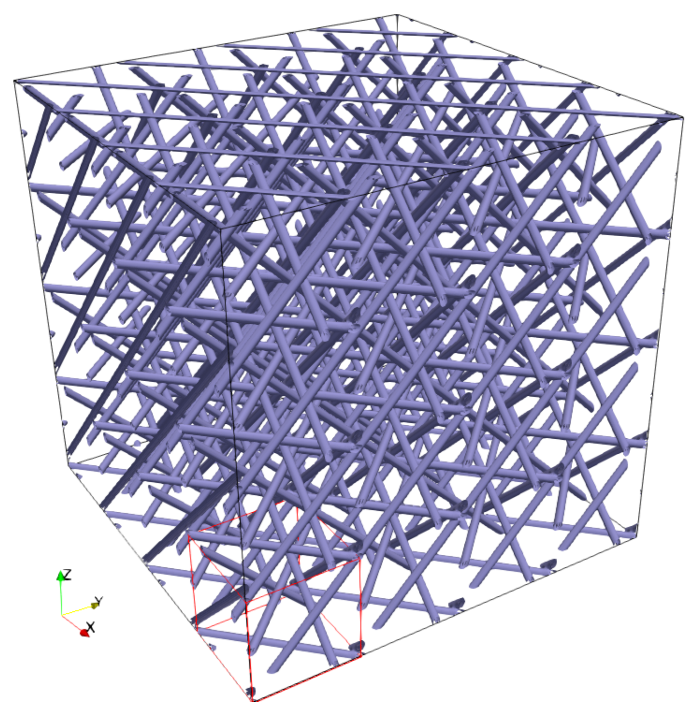
$$\begin{cases} \operatorname{div}_\xi (W(R,\xi) \otimes W(R,\xi)) = 0, \\ \operatorname{div}_\xi W(R,\xi) = 0, \end{cases}$$

$$\fint_{\mathbb{T}^3} W(R,\xi)\, d\xi = 0,$$

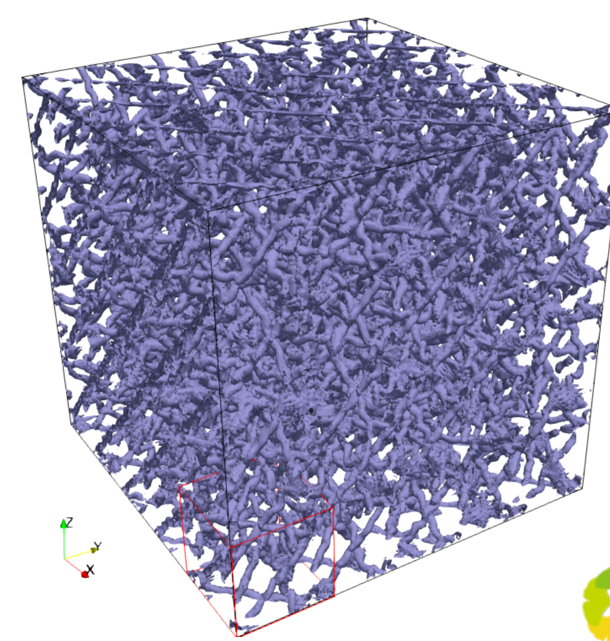
$$\fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi)\, d\xi = R.$$



(a) 1st generation Mikados



(b) 2nd generation Mikados



(c) Ramen: Dynamically evolved Mikados

The slow climb of the Onsager 1/3 Peak (ピーク)

- The convex integration method, of which some highlights have just been described, aims not just to prove the (week) convergence to a solution of the Euler equation. To prove the Onsager conjecture we have to show that the solution can have a spatial Hölder α exponent arbitrarily close (below) to 1/3 and an energy that decreases in time. Very tough!
At first they had $\alpha < 1/10$. Then $\alpha < 1/5$. Then $\alpha < 1/3$ but no control over the energy behaviour.

- The small-scale flow added at each stage had to be static solutions of the 3D Euler equation, with preferably few Fourier harmonics. Beltrami flows, such as ABC seemed plausible candidates. Contrary to Mikados, they are not localised and are unable to "kill" arbitrary Reynolds stress errors. How to best approximate the temporal dynamics was also not clear. Hence it took almost ten years until weak solutions with Kolmogorov 1941/Onsager 1949 scaling were constructed in 2016–2017.

- The present highlights are of course no substitute for reading the original papers by Daneri-Székelyhidi (ARMA 2017), Isett (Annals Math. 2018), Buckmaster-DeLellis-Székelyhidi-Vicol (CPAM, in press). An intermediate version, intended for fluid dynamicists, will probably be written by UF, TM and LS within 6-12 months.

The construction of weak intermittent solutions

(U. Frisch, L. Székelyhidi, T. Matsumoto, S. Modena and others)

- There exist a number of phenomenological models that display intermittency (anomalous scaling of structure functions). They include the β-model, the random β-model, the GOY (Gledzer-Ohkitani-Yamada) model. None of them is derived from the basic Euler or Navier–Stokes equations.

- Superficial inspection ofthe Mikado figures, shown before, suggests that there *might* be something fractal there.
  However, the rules for scaling down the Mikados from one generation (stage) to the next one imply that the fraction of Mikado-active fluid is the same in all generations. This can clearly be modified: we can use the β or random β models to scale down the diameter of the Mikados by more than the factor two used so far.

- This opens the interesting possibility of constructing weak solutions of the Euler equation with bifractal or multifractal scaling. Note that the very concept of multifractality was born from the attempt to model wind-tunnel data, but has otherwise never been directly connected to the equations of high-Reynolds number fluid turbulent fluid flow.

- This issue and many more are at the centre of the international project *Stirring the Turbulence Problem*, soon too be submitted to the Simons Foundation (with participants from Brazil, Canada, France, Germany, India, Italy, Japan, UK, USA).