Shape Optimization for suppressing time periodic flow considering Snapshot POD

o Takashi NAKAZAWA, Osaka university, 1-3, Machikaneyama-cho, Toyonaka-shi, Oaksa, Japan, 560-8531, E-mail: nakazawa@sigmath.es.osaka-u.ac.jp

This paper presents a versatile and Proper Orthogonal Decomposition-based shape optimization method, to delay laminar-turbulent transition and to promote a mixing. The main problem is the nonstationary Navier-Stokes problem, for which time average velocity field is generated to take time integration, and a correction coefficient matrix is formed. Eigenvalues of the correction coefficient matrix is defined as the cost function. Based on Lagrange multiplier method, the objective cost functional is obtained, and by using Adjoint variable method main problem and adjoint problems are solved to evaluate the sensitivity. The two dimensional cavity flow used as an initial domain is reshaped iteratively with H1 gradient method which is able to deform stably.

1. Introduction

A shape optimization method is able to play important roles in flow control, and high speed fluid machinery is designed such as airplane's body, wing and engine inside. Then, to delay laminar-turbulent transition and to promote a mixing are needed.

In spite of the flow stability is playing very important roles in fluid dynamics and fluid engineering, many existing papers have not reported effects of the stability by shape optimization. Thereby, T. Nakazawa [1] reported that the minimizing and maximizing problem of the dissipation energy are solved in the two dimensional cavity flow and the flow stability is changed indirectly by the optimization process, where the stationary Navier-Stokes problem is used as the main problem.

Next, T. Nakazawa and H. Azegami [2] suggested a new pioneering shape optimization method to make the disturbances stable directly, in which a real parts of the leasing eigenvalues is used as the cost function and the stationary Navier-Stokes problem and its eigenvalue problem are defined as the main problems. By the way, T. Nakazawa [3], the author tried to increase the critical Reynolds numbers more by combining the two kind of shape optimization problems. In particular, using the optimal domain obtained in [1] as an initial domain, the minimizing problem of the real part of leading eigenvalues constructed in [2] are solved.

This paper presents a more versatile shape optimization method by using Proper Orthogonal Decomposition (POD) in the view point of "Data Science".

2. The nonstationary Navier-Stokes Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain for $d = 2$, a two dimensional Cavity flow is considered. The initial domain depicts $\Omega_0 \subset \Omega$ in particular

$$\Omega_0 = \Omega_{\mathrm{Full}} \setminus \Omega_{\mathrm{Circ}},$$
$$\Omega_{\mathrm{Full}} = \{\boldsymbol{x} = (x,y) |\ 0 \le x \le 1, 0 \le y \le 1\ \},$$
$$\Omega_{\mathrm{Circ}} = \{\boldsymbol{x} = (x,y) |\ (x-0.5)^2 + (y-0.5)^2 = 0.1^2\ \},$$

$$\partial\Omega_0 = \Gamma_{\mathrm{top}} \cup \Gamma_{\mathrm{wall}}$$

$$\Gamma_{\mathrm{top}} = \{(x,y) |\ 0 \le x \le 1, y = 1\ \}, \Gamma_{\mathrm{wall}} = \partial\Omega_0 \setminus \Gamma_{\mathrm{top}}\,.$$

For one of main problems, the nonstationary Navier-Stokes problem is used,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\boldsymbol{u}, \nabla\cdot\boldsymbol{u} = 0,$$

where velocity vector and pressure are depicted as $\boldsymbol{u} \in U$ and $p \in P$, and Re represents the Reynolds number,

$$U = \left\{\begin{matrix} \boldsymbol{u} \in H^1(\Omega, \mathbb{R}^2); \\ \boldsymbol{u} = (16x^2(x-1)^2, 0)\cos(\pi t) \text{ on } \Gamma_{\mathrm{top}}, \\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{wall}} \end{matrix}\right\},$$

$$P = \left\{p \in L^2(\Omega, \mathbb{R}); \int_\Omega p dx = 0 \text{ in } \Omega\right\}.$$

For all $(\boldsymbol{v}, q) \in V \times P$, the weak form is as follows;

$$\int_\Omega \left\{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\boldsymbol{v} - (\nabla\cdot\boldsymbol{v})q - (\nabla\cdot\boldsymbol{u})p + \nabla\boldsymbol{u}^{\mathrm{T}} : \nabla\boldsymbol{v}^{\mathrm{T}}\right\} dx = 0.$$

where $V = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^2); \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega_0\}$. By discretizing in time with the characteristic curve method, the following weak form to obtain $\{(\boldsymbol{u}^n, p^n)\}_{n=1}^N$ is derived,

$$\int_\Omega \left\{\begin{matrix} \dfrac{\boldsymbol{u}^{n+1}(\boldsymbol{x}) - \boldsymbol{u}^n(\boldsymbol{x} - \Delta t\boldsymbol{u}^n(\boldsymbol{x}))}{\Delta t}\cdot\boldsymbol{v}^{n+1} \\ -(\nabla\cdot\boldsymbol{v}^{n+1})q^{n+1} - (\nabla\cdot\boldsymbol{u}^{n+1})p^{n+1} \\ +\nabla(\boldsymbol{u}^{n+1})^{\mathrm{T}} : \nabla(\boldsymbol{v}^{n+1})^{\mathrm{T}} \end{matrix}\right\} dx = 0,$$

for all $(\boldsymbol{v}^n, q^n) \in V \times P$ with $n \in [1, N]$ and the time step $N \in \mathbb{N}$. By the way, the solution of the stationary Navier-stokes problem is chosen as the initial condition $(\boldsymbol{u}^0, p^0)$.

3. Proper Orthogonal Decomposition

At first, taking time integration for the nonstationary Navier-Stokes problem from T_1 to T_2. Next, correlation coefficient matrix $R(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) \in \mathbb{R}^{m\times m}$ is formed where Δt depicts the time step size, and

$$R(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}) = \int_\Omega \widetilde{\boldsymbol{u}}^{\mathrm{T}}\widetilde{\boldsymbol{u}} dx,$$

$$m = N_2 - N_1 + 1, T_1 = \Delta t N_1, T_2 = \Delta t N_2$$

Second Proper Orthogonal Decomposition (POD) is used to obtain eigenvalues $\omega \in \mathbb{R}^m$ and functions $\hat{\boldsymbol{u}} \in \mathbb{R}^{m\times m}$ by solving eigenvalue problem $R(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}})\hat{\boldsymbol{u}} = \omega\hat{\boldsymbol{u}}$. Finally, we have POD basis $\boldsymbol{\Phi} = \omega^{-\frac{1}{2}}\hat{\boldsymbol{u}}\tilde{\boldsymbol{u}} \in \mathbb{R}^{m\times m}$.

4. Shape Optimization Problem

In the shape optimization problem, the first variation of the functional $\dot{L}(\boldsymbol{\phi})$ is the same as the material derivative;

$$\dot{L}(\boldsymbol{\phi}, \Omega \cup \Gamma_{\mathrm{wall}}) = \int_\Omega G'(\boldsymbol{x}, u, \nabla u)\, dx + \int_{\partial\Omega} G(\boldsymbol{x}, u, \nabla u)\boldsymbol{\nu}\cdot\boldsymbol{\varphi}\, d\gamma,$$

where $(\cdot)'$ represents the shape derivative and $\boldsymbol{\nu}$ depicts the outward normal vector on the boundary. And more the domain variation is defined as $\boldsymbol{\phi} = \boldsymbol{\phi_0} + \boldsymbol{\epsilon\varphi} + \boldsymbol{o}(\boldsymbol{\epsilon}^2)$, where $\boldsymbol{\phi_0}$ depicts the identify map and the function space for $\boldsymbol{\phi} \in D$ is

$$D = \left\{ \begin{matrix} \boldsymbol{\phi} \in W^{1,\infty}(\Omega, \mathbb{R}^2); \\ \|\boldsymbol{\phi} - \boldsymbol{\phi_0}\|_{W^{1,\infty}(\Omega,\mathbb{R}^2)} < 1, \overline{\boldsymbol{\phi}(\Omega)} \subset \Omega \end{matrix} \right\}.$$

The main problems are the nonstationary Navier-Stokes problem, and an eigenvalue problem of POD. For convenience, we define the following functional;

$$L_1(\boldsymbol{\phi}) = \frac{1}{m}\sum_{n=N_1}^{N_2} \int_{\boldsymbol{\phi}(\Omega)} G_1(x, \boldsymbol{u}^n, p^n, \boldsymbol{v}^n, q^n)\, dx,$$

$$\begin{aligned} G_1(x, \boldsymbol{u}^n, p^n, \boldsymbol{v}^n, q^n) &= \frac{\boldsymbol{u}^{n+1}(\boldsymbol{x}) - \boldsymbol{u}^n(\boldsymbol{x} - \Delta t\boldsymbol{u}^n(\boldsymbol{x}))}{\Delta t}\cdot \boldsymbol{v}^{n+1} \\ &\quad - (\nabla\cdot\boldsymbol{v}^{n+1})q^{n+1} - (\nabla\cdot\boldsymbol{u}^{n+1})p^{n+1} \\ &\quad + \nabla(\boldsymbol{u}^{n+1})^{\mathrm{T}} : \nabla(\boldsymbol{v}^{n+1})^{\mathrm{T}}. \end{aligned}$$

For $\boldsymbol{\alpha} \in \mathbb{R}^{m\times m}$ and $\delta_{j\to k}$ for extracting from the j primary component to the k primary component,

$$L_2(\boldsymbol{\phi}) = \frac{1}{m}\sum_{n=N_1}^{N_2} \delta_{j\to k} G_2(x, \omega, \hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{\alpha}),$$

$$G_2(x, \omega, \hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{\alpha}) = \left\{\omega\hat{\boldsymbol{u}} - \left(\int_{\boldsymbol{\phi}(\Omega)} \tilde{\boldsymbol{u}}^{\mathrm{T}}\tilde{\boldsymbol{u}} dx\right)\hat{\boldsymbol{u}}\right\}\boldsymbol{\alpha}$$

Based on the above preparations of the functional, the sum of the eigenvalues $f(\boldsymbol{\phi})$ is defined as the cost function. So, the objective functional is represented by

$$L(\phi) = f(\boldsymbol{\phi}) - L_1(\boldsymbol{\phi}) - L_2(\boldsymbol{\phi}),$$
$$f(\boldsymbol{\phi}) = \frac{1}{m}\sum_{n=N_1}^{N_2} \delta_{j\to k}\omega^n.$$

Anyway, in the view point of Lagrange multiplier method, the trial function $(\boldsymbol{v}^n, q^n) \in V \times P$ to solve $L_1(\boldsymbol{\phi}) = 0$ with FEM is the same as the multiplier of the objective functional $L(\phi)$, and $\boldsymbol{\alpha} \in \mathbb{R}^{m\times m}$ for $\hat{\boldsymbol{u}}$.

Based on Lagrange multiplier method, we can derive the main problem (Section 2 and Section 3) and the adjoint problems. From the main problem, it is possible to obtain the solutions $\{(\boldsymbol{u}^n, p^n)\}_{n=1}^N$ and $(\omega, \hat{\boldsymbol{u}})$ for the nonstationary Navier-Stokes problem and the eigenvalue problem for POD. Especially, from the adjoint problems we should solve the following partial differential equation;

$$(\nabla\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \nabla q - \frac{1}{\mathrm{Re}}\Delta\boldsymbol{v} = \overline{2\sqrt{\omega}\boldsymbol{\Phi}\hat{\boldsymbol{u}}},$$
$$\nabla\cdot\boldsymbol{v} = 0.$$

Finally, by substituting the main variables $\{(\boldsymbol{u}^n, p^n)\}_{n=1}^N$ and $(\omega, \hat{\boldsymbol{u}})$ and the adjoint variables $(\boldsymbol{v}, q)$ and $\boldsymbol{\alpha}$ into the first variation of the functional, and we can obtain the first variation as

$$\dot{L}(\boldsymbol{\phi}) = \int_{\Gamma_{\mathrm{wall}}} \left\{\frac{1}{m}\sum_{n=N_1}^{N_2}\frac{1}{\mathrm{Re}}\{\nabla(\boldsymbol{u}^n)^{\mathrm{T}} : \nabla(\boldsymbol{v}^n)^{\mathrm{T}}\}\boldsymbol{\nu}\right\}\cdot\boldsymbol{\varphi}\, d\gamma,$$

and the sensitivity is evaluated by

$$\boldsymbol{\varphi} = \int_{\Gamma_{\mathrm{wall}}} \left\{\frac{1}{m}\sum_{n=N_1}^{N_2}\frac{1}{\mathrm{Re}}\{\nabla(\boldsymbol{u}^n)^{\mathrm{T}} : \nabla(\boldsymbol{v}^n)^{\mathrm{T}}\}\boldsymbol{\nu}\right\} d\gamma.$$

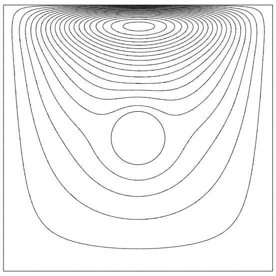
For numerical calculations, we should smooth $\boldsymbol{\varphi}$ with H^1 gradient method, because the sensitivity is in $L^\infty(\Omega, \mathbb{R}^2)$ and lack of the smoothness.

5. Numerical Results

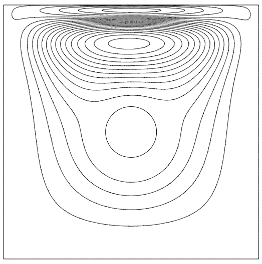
The suggested shape optimization problem is demonstrated at Re=100. For the test calculation, POD is performed to show the stream lines for POD basis and eigenvalues in Fig. 1. Next, I investigated numerical accuracy by increasing the number of elements and vertices depending NN. Fig. 2 shows ω_2 with the reshaping steps based on NN, and the numerical result is asymptotic at NN=130. Finally, I operate the shape optimization problem changing $\delta_{j\to k}$, and $\delta_{1\to m}$ gives us the smaller eigenvalues ω_2 for the second primary component than the shape optimization defining the dissipation energy as the cost function. Fig. 4 explains the optimal shape for $\delta_{1\to m}$.

6. Numerical Results

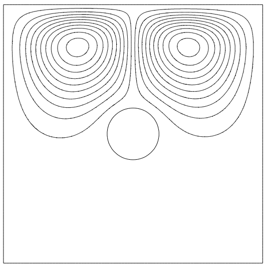
In this study, the shape optimization method for flow stability control is suggested, where the cost function is defined as the sum of the eigenvalues based on POD and the nonstationary Navier-Stokes problem is used as the main problem. $\delta_{1\to m}$ gives us the smaller eigenvalues ω_2 for the second primary component than the shape optimization defining the dissipation energy as the cost function.



(a) The first primary component ($\omega_1 = 0.151706$)



(b) The second primary component ($\omega_2 = 0.0221411$)



(c) The third primary component ($\omega_3 = 0.00582061$)

Fig. 1. Stream lines for POD basis and eigenvalues.

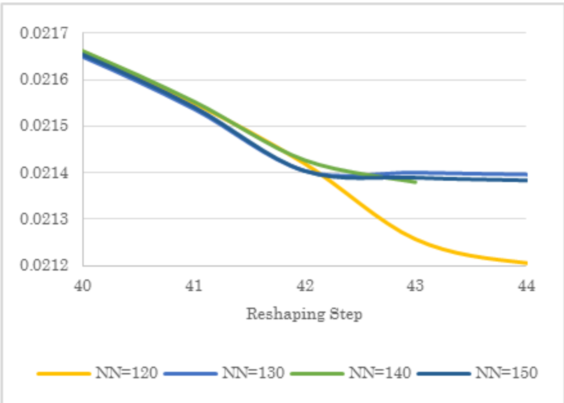


Fig. 2. ω_2 with the reshaping steps based on NN.

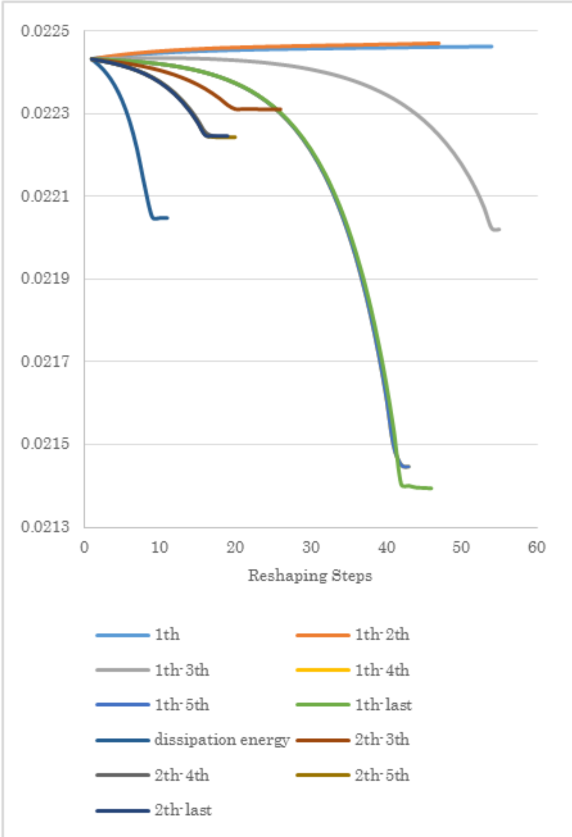


Fig. 3. ω_2 with the reshaping steps based on $\delta_{j\to k}$.

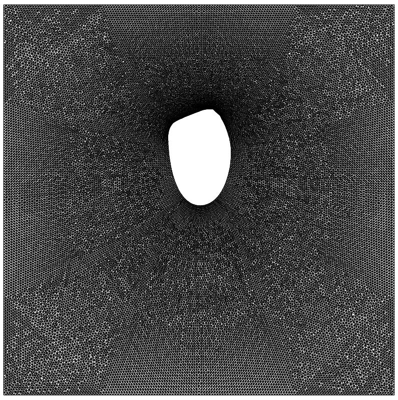
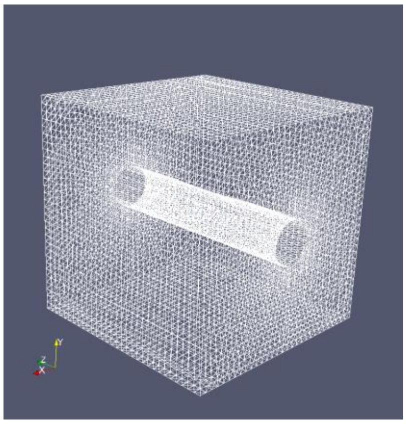
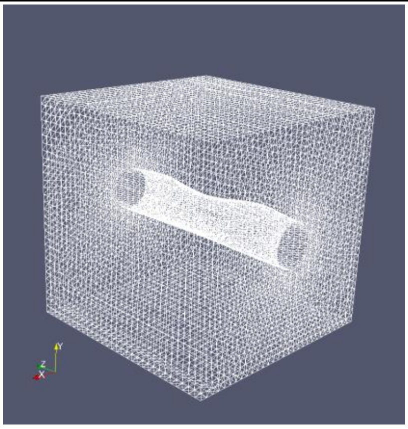


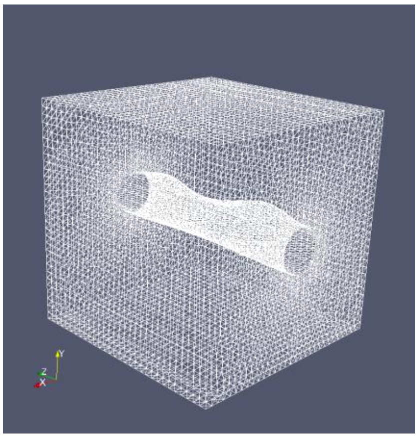
Fig. 4. The optimal shape for $\delta_{1\to m}$.



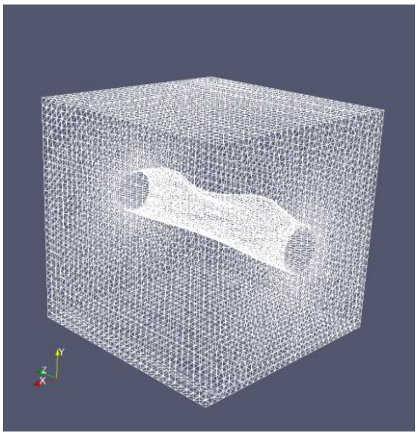
(a)



(b)



(c)



(d)

Fig. 5. The shape improving in 3 dimensional domain.

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