Hermite bicubic, quartic and quintic stream functions for Incompressible flows in two dimensions

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1. Introduction

This paper describes a recent development of divergence-free basis functions based on Hermite interpolated stream functions. First, the incompressible Navier-Stokes equation(INS) is orthogonally decomposed into a solenoidal part and an irrotational part by the Helmholtz theorem such that

$$\frac{\partial}{\partial t}\underline{u} = \pi^S(-\underline{u}\cdot\nabla\underline{u} + \nu\nabla^2\underline{u}) + \underline{f}^S \quad (1a)$$

$$\frac{1}{\rho}\nabla p = \pi^I(-\underline{u}\cdot\nabla\underline{u} + \nu\nabla^2\underline{u}) + \underline{f}^I \quad (1b)$$

Second, the equations (1a) and (1b) are projected onto corresponding solution fields using weighting functions $\underline{v}$ and $\underline{w}$ from solenoidal and irrotational functions, respectively:

$$(\underline{v}, \frac{\partial}{\partial t}\underline{u}) = -(\underline{v}, \underline{u}\cdot\nabla\underline{u}) - \nu(\nabla\underline{v}, \nabla\underline{u}) + \nu\oint \underline{v}\nabla\underline{u}\cdot\hat{n}d\Gamma \quad (2a)$$

$$(\underline{w}, \nabla p) = -(\underline{w}, \underline{u}\cdot\nabla\underline{u}) + \nu(\underline{w}, \nabla^2\underline{u}) + (\underline{w}, \underline{f}^I) \quad (2b)$$

2. Interpolating functions on a quadrilateral

Irrotational components of a scalar potential and solenoidal components of a vector potential are explained. The scalar potential within an element is evaluated as, for bicubic,

$$\phi^e(x,y) = \sum_{i=1}^{n} \underline{g}_i^T \underline{\Phi}_i^e = \sum_{i=1}^{n} \begin{pmatrix} g_0 & g_x & g_y & g_{xy} \end{pmatrix}_i \underline{\Phi}_i^e$$

and the vector potential is approximated by

$$\psi^e(x,y) = \sum_{i=1}^{n} \underline{s}_i^T \underline{\Psi}_i^e = \sum_{i=1}^{n} \begin{pmatrix} g_0 & g_y & -g_x & g_{xy} \end{pmatrix}_i \underline{\Psi}_i^e$$

The irrotational and the solenoidal basis functions are explained.

3. Finite Element Formulations

The discrete counterparts of the variational equations (2a) and (2b) are used to formulate finite element approximations. From a finite dimensional space spanned by the solenoidal and the irrotational functions, one can compose matrices within an element. We show how to construct element matrices such as mass, convection and diffusion matrices.

4. Numerical Study

We present numerical results from three benchmark problems. The first problem is the lid-driven cavity problem suggested by Ghia, et al. The second problem is the backward-facing step problem suggested by Gartling for stability studies in a channel flow. The third one is a buoyancy-driven flow within a square cavity.

In the lid-driven cavity case, for Re = 1000, we confirm accuracy by comparison with a published report by Botella, et al. For Re = 3200, vorticity contours are shown. One may find that the shape of the vorticity contours are very much alike the one of Ghia, et al. For Re = 10000, streamlines are shown for the three methods: S3416, S4424 and S5424. All methods showed well-developed secondary flows at corners similar to the benchmark solutions.

In the backward-facing step problem, we employed a simplified version suggested by Gartling. The gid used are from 120 x 8 elements, and numerical values are compared at x=7 with those by Gartling. S3416 and S4424 have shown a good trend in the horizontal component of velocity, and S3416 has shown a good trend in the vertical velocity component. Streamlines and pressure contours from S3416 and S4424 are shown. Both methods show well-developed two re-circulating regions at Re = 800.

In the buoyancy-driven flow, we have shown that the Hermite stream function method can be extended to this coupled problem. Governing equations are presented in dimensionless form, and finite element formulations are developed. From the discretized equations, decoupled solution procedure has been introduced within the Newton correction iterations. Numerical values are compared with benchmark solutions suggested by De Vahl Davis for Rayleigh numbers for 10^3 ~ 10^6 using Pr = 0.71. Contour plots for stream function, temperature and vorticity are shown at Ra = 10^6. The contour plots are in good agreements with those by Davis. Considering that the present results are from 32 x 32 uniform elements, the performances of the Hermite stream function method is very affirmative.

5. Summary

It has been shown that Hermite functions can be used efficiently to construct velocity basis functions that are divergence-free. Considering that the present algorithm does not require an up-winding or a stabilization terms, the present algorithm should be recognized in its accuracy and stability.